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From Kinetic to Navier-Stokes-Type Equations

M. LACHOWICZ
Department of Mathematics, Warsaw University
Warsaw, Poland
lachowic@mimuw.edu.pl

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Abstract—Some kinetic equations with symmetrized kernels in a box are considered. It is shown that, for a particular choice of relations between the small parameters of the models, the equations result in the Navier-Stokes-type system for compressible fluids at the 0^{th} order approximation. The viscosity and heat conduction terms, in the resulting system, are independent of the small parameters. It is proved that the existence of smooth solutions of the Navier-Stokes-type system implies both existence of solutions to the kinetic equations and asymptotic relationship.

Keywords—Kinetic equations, Navier-Stokes system, Singularly perturbed problem, Hydrodynamic limit.

The Boltzmann equation in the hydrodynamic limit, as the Knudsen number ε tends to 0 (and the Mach number remains fixed), is consistent with the compressible Navier-Stokes system, with the viscosity and heat conduction terms of the order of ε (cf. [1–3] and references therein). On the other hand, if both the Knudsen number ε and the Mach number $\mathfrak{m}$ are of the same order and tend to 0, the Boltzmann equation is consistent with the incompressible Navier-Stokes equation for $O(\varepsilon)$-perturbation of a global equilibrium (see [4–6]).

On the contrary, one cannot expect to find an asymptotic relation between the Boltzmann equation and the compressible Navier-Stokes system with viscosity and heat conduction terms which are $O(1)$ in the hydrodynamic limit $\varepsilon \downarrow 0$.

The aim of this paper is to show that some kinetic equations can result in a compressible Navier-Stokes-type system, with viscosity and heat conduction terms which are independent of all small parameters of the kinetic models. The approximation is attained using a particular assumption about the relation between the small parameters. The existence of smooth solutions of the Navier-Stokes-type system implies both existence of solutions to the kinetic equations and asymptotic relationship.

Consider the Boltzmann-Enskog equation (B-EE) with a symmetrized kernel

$$Df = \frac{1}{\varepsilon} E_R(f), \tag{1}$$

where $D = \partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}$ is the free-streaming operator, and E_R is the following collision operator:

$$\begin{aligned} E_R(f)(t,\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} & \left(f\left(t,\mathbf{x}+R\mathbf{n},\mathbf{w}'\right) f\left(t,\mathbf{x},\mathbf{v}'\right)\right. \\ & \left. -f(t,\mathbf{x}-R\mathbf{n},\mathbf{w}) f(t,\mathbf{x},\mathbf{v})\right) \left|\mathbf{n}\cdot(\mathbf{w}-\mathbf{v})\right| \,d\mathbf{n}\,d\mathbf{w}, \end{aligned}$$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

$\mathbb{S}^2 = \{\mathbf{n} \in \mathbb{R}^3 : |\mathbf{n}| = 1\}$, $\mathbf{v}' = \mathbf{v} + ((\mathbf{w}-\mathbf{v})\cdot\mathbf{n})\mathbf{n}$, $\mathbf{w}' = \mathbf{w} - ((\mathbf{w}-\mathbf{v})\cdot\mathbf{n})\mathbf{n}$ and the dimensionless parameter $0 < R < \infty$ represents the scale of diameter of the particles.

B-EE is obtained from the Enskog equation (cf. [7]) by putting the high-density factor Y equal to 1. The symmetrization of the kernel in B-EE was introduced by Arkeryd [8]. Such an equation was studied in [9,10].

The Povzner equation (PE) [3,11] reads

$$Df = \frac{1}{\varepsilon} P_R(f), \tag{2}$$

$$P_R(f)(t,\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^3}\int_0^R \frac{(3\alpha^2)}{R^3}\int_{\mathbb{S}^2} \left(f\left(t,\mathbf{x}+\alpha\mathbf{n},\mathbf{w}'\right) f\left(t,\mathbf{x},\mathbf{v}'\right)\right.$$
$$\left. - f\left(t,\mathbf{x}-\alpha\mathbf{n},\mathbf{w}\right) f\left(t,\mathbf{x},\mathbf{v}\right)\right) B(\mathbf{n},\mathbf{w}-\mathbf{v})\,d\mathbf{n}\,d\alpha\,d\mathbf{w},$$

$0 < R < \infty$. Throughout this paper, it is assumed that the collision kernel B corresponds to Grad's cutoff hard potential (the hard-sphere potential is included) and is symmetrized

$$B(-\mathbf{n},\mathbf{w}-\mathbf{v}) = B(\mathbf{n},\mathbf{w}-\mathbf{v}) = B\left(\mathbf{n},\mathbf{w}'-\mathbf{v}'\right), \qquad \forall\,\mathbf{n}\in\mathbb{S}^2, \quad \forall\,\mathbf{v},\mathbf{w}\in\mathbb{R}^3. \tag{3}$$

PE allows for a spatial "smearing" of the process of collisions. In this paper, the smearing on a ball with radius R is assumed.

B-EE (1) and PE (2) have some nice mathematical properties, which are unknown for the Boltzmann equation. The existence of unique global solutions and some their regularity for the B-EE was proved by Arkeryd [12]. The nature of PE is even better—it preserves some properties of the space homogeneous Boltzmann equation. In particular, the existence of unique smooth global solutions is known [11,13]. In the limit $R \downarrow 0$, the solution of the B-EE converges to a renormalized solution of the Boltzmann equation [14]. The same certainly should be true for PE.

Both equations (1) and (2) in the hydrodynamic limit $\varepsilon \downarrow 0$ result in the same hydrodynamic systems and methods of analysis are exactly the same. In this paper, the analysis is performed only for PE, but the results refer to both the equations.

The equation $P_R(f,f) = 0$ for fixed $R > 0$ has a class of solutions too small to generate reasonable (nontrivial) hydrodynamics. Therefore, equation (2) is considered in the limit

$$\varepsilon \downarrow 0 \quad \text{and} \quad R \downarrow 0. \tag{4}$$

Different relations between the small parameters ε and R can lead to different hydrodynamic equations. It is natural to assume that $R \sim \varepsilon^p$, $p > 0$. The most interesting case is $p = 1/2$, for which equation (2) results in the Navier-Stokes-type system at the 0^{th} order approximation.

Formally, in the limit (4), equation (2) leads to a solution in the form of a Maxwellian

$$M[\varrho,\mathbf{u},T](t,\mathbf{x},\mathbf{v}) = \varrho(t,\mathbf{x})\left(2\pi T(t,\mathbf{x})\right)^{-3/2}\exp\left(-\frac{|\mathbf{v}-\mathbf{u}(t,\mathbf{x})|^2}{2T(t,\mathbf{x})}\right),$$

for some fluid-dynamic parameters ϱ, $\mathbf{u}$, T.

Assume that ϱ, $\mathbf{u}$, T are known, smooth, and that ϱ together with T are positive. Consider the space $L_2\{M\}$ equipped with the norm $\|f; L_2\{M\}\| = \|fM^{-(1/2)}; L_2\mathbb{R}^3)\|$, and with the inner product $(f_1,f_2)_{L_2\{M\}} = (f_1M^{-1}, f_2)_{L_2(\mathbb{R}^3)}$.

Define in $L_2\{M\}$ the "hydrodynamic" and "nonhydrodynamic" subsets $\mathcal{N} = \operatorname{lin}\{M\psi_i : i = 0,\dots,4\}$ and $\mathcal{R} = \mathcal{N}^\perp = \{f \in L_2\{M\} : (f, M\psi_i)_{L_2\{M\}} = 0;\ i = 0,\dots,4\}$, where $\psi_0,\dots,\psi_4$ are the collision invariants $\psi_0 \equiv 1$, $\psi_i(\mathbf{v}) = v_i\ (i = 1,2,3)$, $\psi_4(\mathbf{v}) = |\mathbf{v}|^2$.

Define in $L_2\{M\}$ the projection operators $\mathcal{P}$ and $\mathcal{P}^\perp = 1 - \mathcal{P}$ onto $\mathcal{N}$ and $\mathcal{R}$, respectively.

In the Hilbert procedure, the solution is searched in the form $f = \sum_{j=0} \varepsilon^j f^{(j)}$ where $f^{(j)} = g^{(j)} + h^{(j)}$ for $j \geq 1$ and $g^{(j)} \in \mathcal{R}$, $h^{(j)} \in \mathcal{N}$. This leads to the following set of equations:

$$f^{(0)} = M[\varrho, \mathbf{u}, T], \qquad \mathcal{P} D f^{(0)} - \frac{R^2}{\varepsilon} \mathcal{P} P_0^{(2)} \left(f^{(0)}, f^{(0)}\right) = 0, \tag{5}$$

$$2J\left(f^{(0)}, g^{(1)}\right) = \mathcal{P}^{\perp} D f^{(0)} - \frac{R^2}{\varepsilon} \mathcal{P}^{\perp} P_0^{(2)} \left(f^{(0)}, f^{(0)}\right), \tag{6}$$

$$\dots;$$

where $P_0^{(i)}$ is the symmetrization of the operator $\tilde{P}_0^{(i)}$,

$$\begin{aligned}\tilde{P}_0^{(i)}(f,f)(t,\mathbf{x},\mathbf{v}) = \frac{1}{i!} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} &\left(\left((\mathbf{n} \cdot \partial_{\mathbf{x}})^i f \right)(t,\mathbf{x},\mathbf{w}') f(t,\mathbf{x},\mathbf{v}') \right. \\ &\left. - \left((-\mathbf{n} \cdot \partial_{\mathbf{x}})^i f \right)(t,\mathbf{x},\mathbf{w}) f(t,\mathbf{x},\mathbf{v}) \right) B(\mathbf{n}, \mathbf{w} - \mathbf{v}) \, d\mathbf{n} \, d\mathbf{w},\end{aligned}$$

and $J = P_0^{(0)}$ is the Boltzmann collision operator. Clearly, $P_0^{(i)}(f_1, f_2) = 0$, for each odd integer i (and f_1, f_2, for which the integrals make sense). This property certainly is not true for operators without the symmetrization of the kernel.

For $R = \sqrt{b\,\varepsilon}$, equation (5) leads to

$$\partial_t \varrho + \partial_{x_i}(\varrho u_i) = 0, \tag{7a}$$

$$\partial_t(\varrho u_j) + \partial_{x_i}(\varrho u_i u_j) + \partial_{x_j}(\varrho T) = b\, \partial_{x_i}\left(\varrho^2 \mu_{ijkl}(T) \partial_{x_k} u_l\right), \qquad j = 1,2,3, \tag{7b}$$

$$\begin{aligned}\partial_t \left(\varrho \left(\frac{3}{2} T + \frac{1}{2} |\mathbf{u}|^2 \right) \right) + \partial_{x_i} \left(\varrho u_i \left(\frac{3}{2} T + \frac{1}{2} |\mathbf{u}|^2 \right) + \varrho u_i T \right) \\ = b \left(\partial_{x_i}\left(\varrho^2 u_j \mu_{ijkl}(T) \partial_{x_k} u_l\right) + \partial_{x_i}\left(\varrho^2 \mu_{ijk}^H(T) \partial_{x_k} T\right)\right),\end{aligned} \tag{7c}$$

where

$$\begin{aligned}\mu_{ijkl}(T) &= \frac{1}{\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} n_i n_j n_k \, \mathbf{n} \cdot (\mathbf{w} - \mathbf{v}) B\left(\mathbf{n}, \sqrt{2T}(\mathbf{w} - \mathbf{v})\right) \\ &\qquad \times (w_l - v_l) \exp\left(-|\mathbf{w}|^2 - |\mathbf{v}|^2\right) d\mathbf{n} \, d\mathbf{w}, \\ \mu_{ijk}^H(T) &= \frac{1}{2\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} n_i n_j n_k \, \mathbf{n} \cdot (\mathbf{w} - \mathbf{v}) \mathbf{n} \cdot (\mathbf{w} + \mathbf{v}) B\left(\mathbf{n}, \sqrt{2T}(\mathbf{w} - \mathbf{v})\right) \\ &\qquad \times \left(|\mathbf{w}|^2 - |\mathbf{v}|^2\right) \exp\left(-|\mathbf{w}|^2 - |\mathbf{v}|^2\right) d\mathbf{n} \, d\mathbf{w}, \\ \mathbb{S}_+^2 &= \{\mathbf{n} \in \mathbb{R}^3 : |\mathbf{n}| = 1, \ \mathbf{n} \cdot (\mathbf{w} - \mathbf{v}) > 0\},\end{aligned}$$

b is assumed to be a fixed positive number, and repeated indices, according to Einstein's convection, are understood summed from 1 to 3.

System (7) is the Navier-Stokes-type system of equations of the theory of compressible fluids (cf. [15,16]). Its viscosity and heat conduction terms are independent of the small parameters of kinetic equations (2) (or (1)).

Of special interest is system (7) in the case of the collision kernel B corresponding to the hard sphere potential. In this case, the system reads (the integrals are calculated in [17, Appendix D]),

$$\partial_t \varrho + \partial_{x_i}(\varrho u_i) = 0, \tag{8a}$$

$$\begin{aligned}\partial_t(\varrho u_j) + \partial_{x_i}(\varrho u_i u_j) + \frac{\partial}{\partial x_j}(\varrho T) = \frac{8\sqrt{\pi}}{15} b \left(\partial_{x_i}\left(\varrho^2 \sqrt{T}\left(\partial_{x_i} u_j + \partial_{x_j} u_i\right)\right) \right. \\ \left. + \partial_{x_j}\left(\varrho^2 \sqrt{T} \partial_{x_k} u_k\right)\right), \qquad j = 1,2,3,\end{aligned} \tag{8b}$$

$$\begin{aligned}\partial_t \left(\varrho\left(\frac{3}{2}T+\frac{1}{2}|\mathbf{u}|^2\right)\right) + \partial_{x_i}\left(\varrho u_i\left(\frac{3}{2}T+\frac{1}{2}|\mathbf{u}|^2\right)+\varrho u_i T\right)\\ = \frac{8\sqrt{\pi}}{15}\, b\left(\partial_{x_i}\left(\varrho^2\sqrt{T}u_j\left(\partial_{x_i}u_j+\partial_{x_j}u_i\right)\right)\right.\\ \left.+\partial_{x_j}\left(\varrho^2\sqrt{T}u_j\partial_{x_k}u_k\right)+\frac{5}{2}\partial_{x_i}\left(\varrho^2\sqrt{T}\partial_{x_i}T\right)\right).\end{aligned} \tag{8c}$$

Throughout the paper, the equations are considered in a periodic box (i.e., on the three-dimensional torus $\mathbb{T}^3$) with respect to the space variable $\mathbf{x}$.

ASSUMPTION 1. *Let ϱ, u_i, $T : [0,t_0]\times\mathbb{T}^3\to\mathbb{R}^1$, $i=1,2,3$, be the smooth functions, for some $t_0\in]0,+\infty[$, such that $\varrho(t,\mathbf{x})>c_1>0$, $T(t,\mathbf{x})>c_2>0$, $\forall t\in[0,t_0]$, $\forall\mathbf{x}\in\mathbb{T}^3$, where c_1 and c_2 are constants (independent of ε).*

Let $M=M[\varrho,\mathbf{u},T]$, where ϱ, $\mathbf{u}$, T are as in Assumption 1 and let $M_0=M_{t=0}$.

A simple consequence of Assumption 1 is the existence, for each $\alpha\in\mathbb{R}^1$, of positive constants c_α^-, c_α^+, T_-, and T_+—independent of ε and such that

$$c_\alpha^-\omega_-(\mathbf{v})\le(1+|\mathbf{v}|)^\alpha M(t,\mathbf{x},\mathbf{v})\le c_\alpha^+\omega_+(\mathbf{v}),\qquad\forall t\in[0,t_0],\quad\forall\mathbf{x}\in\mathbb{T}^3,\quad\forall\mathbf{v}\in\mathbb{R}^3, \tag{9}$$

where ω_- and ω_+ are the global Maxwellians $\omega_-=M[1,\mathbf{0},T_-]$, $\omega_+=M[1,\mathbf{0},T_+]$.

Let $\mathbb{Y}_0^{\alpha,k}$ and $\mathbb{Y}_+^{\alpha,k}$ be the spaces equipped with the norms (cf. [3]),

$$\|f\|_0^{\alpha,k}=\sup_{\substack{0\le|\gamma|\le k\\ \mathbf{x}\in\mathbb{T}^3\\ \mathbf{v}\in\mathbb{R}^3}}\left|(1+|\mathbf{v})^\alpha M_0^{-1/2}(\mathbf{x},\mathbf{v})\frac{\partial^{|\gamma|}f}{\partial\mathbf{x}^\gamma}(\mathbf{x},\mathbf{v})\right|,$$

$$\|f\|_+^{\alpha,k}=\sup_{\substack{0\le|\gamma|\le k\\ \mathbf{x}\in\mathbb{T}^3\\ \mathbf{v}\in\mathbb{R}^3}}\left|(1+|\mathbf{v}|)^\alpha\omega_+^{-1/2}(\mathbf{x},\mathbf{v})\frac{\partial^{|\gamma|}f}{\partial\mathbf{x}^\gamma}(\mathbf{x},\mathbf{v})\right|,$$

respectively.

By the methods of [3,9], one obtains the following theorem.

THEOREM 1. *Let $R=\sqrt{b\,\varepsilon}$ (for b—a fixed positive number) and let $t_0\in]0,+\infty[$ be such that on the time interval $[0,t_0]$, there exists a solution $(\varrho,\mathbf{u},T)$ of system (8) such that Assumption 1 is satisfied. Moreover, let the initial data be such that $f|_{t=0}=M_0+G$, where $M_0=M[\varrho|_{t=0},\mathbf{u}|_{t=0},T|_{t=0}]$, and G satisfies*

$$\|G\|_0^{0,4}\le\kappa,\qquad\int G\,d\mathbf{v}=\int v_iG\,d\mathbf{v}=\int|\mathbf{v}|^2G\,d\mathbf{v}=0,\qquad i=1,2,3,\qquad G\in\mathbb{Y}_0^{\alpha,k},$$

with α and k being large enough, where κ is a critical constant (independent of ε).

If $0<\varepsilon\le\varepsilon_0$, where ε_0 is a critical value (depending on t_0), then a solution f of equation (2) exists in $f\in C^0([0,t_0];\mathbb{Y}_+^{\alpha_0,k_0})\cap C^1(]0,t_0[;\mathbb{Y}_+^{\alpha_0,k_0-1})$, for some $\alpha_0>0$, $k_0\ge1$ and

$$\sup_{0\le t\le t_0}\left\|f(t)-M[\varrho,\mathbf{u},T](t)-\tilde{f}^{(0)}\left(\frac{t}{\varepsilon}\right)\right\|_+^{\alpha_0,k_0}\le c_{t_0}\varepsilon, \tag{10}$$

where c_{t_0} is a constant (depending on t_0) and $\tilde{f}^{(0)}\in C^0\left([0,t_0[;\mathbb{Y}_0^{\alpha_1,k_1}\right)$ is such that

$$\left\|\tilde{f}^{(0)}(\tau)\right\|_0^{\alpha_1,k_1}\le c_Ge^{-\delta\tau}, \tag{11}$$

for some $\delta>0$, $\alpha_1>0$, and $k_1>0$; c_G is a constant depending on G and $c_0=0$.

PROOF. The proof follows the line reviewed in [3]. The crucial point is that the linearization $L_R f = 2M^{-1/2} P_R(M, M^{1/2} f)$ of the operator P_R can be split $L_R = \bar{L}_R + R^2 A_R$, where the operator $\bar{L}_R$ is nonpositive in $L_2(\mathbb{T}^3 \times \mathbb{R}^3)$ and the operator A_R has "good" estimations (cf. [9, Lemma 6.1]). ∎

REMARKS.

1. The analogous theorem holds if equation (2) is replaced by equation (1).
2. For $R = \varepsilon$, a theorem similar to Theorem 1 may be proved which differs in that both equations (1) and (2) result in the classical Euler system of the theory of compressible fluids, as well as the Navier-Stokes system, with both the viscosity and heat conduction terms proportional to ε (see [3]) rather than system (7)—exactly as in the case of the Boltzmann equation.

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